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CS109

Handout #4
March 30, 2016

Problem Set #1
Due: 2:30pm on Friday, April 8th

For each problem, briefly explain/justify how you obtained your answer. Brief explanations of your answer are necessary to get full credit for a problem even if you have the correct numerical answer. The explanations help us determine your understanding of the problem whether or not you got the correct answer. Moreover, in the event of an incorrect answer, we can still try to give you partial credit based on the explanation you provide. It is fine for your answers to include summations, products, factorials, exponentials, or combinations; you don't need to calculate those all out to get a single numeric answer.

Note: all assignment submissions will be made online. If you handwrite your solutions, you are responsible for making sure that you can produce clearly legible scans of them for submission. You may use any word processing software you like for writing up your solutions. On the CS109 webpage we provide a template file and tutorial for the LaTex system, if you'd like to use it.

1. How many ways can 12 people be seated in a row if
    a. there are no restrictions on the seating arrangement?
    b. persons A and B must sit next to each other?
    c. there are 6 men and 6 women and no two men nor two women can sit next to each other?
    d. there are 5 men and they must sit next to each other?
    e. there are 6 married couples and each couple must sit together?

2. 9 computers are brought in for servicing (and machines are serviced one at a time). Of the 9 computers, 3 are PCs, 4 are Macs, and 2 are Linux machines. Assume that all computers of the same type are indistinguishable (i.e., all the PCs are indistinguishable, all the Macs are indistinguishable, etc.).
    a. In how many distinguishable ways can the computers be ordered for servicing?
    b. In how many distinguishable ways can the computers be ordered if the first 5 machines serviced must include all 4 Macs?
    c. In how many distinguishable ways can the computers be ordered if 2 PCs must be in the first three and 1 PC must be in the last three computers serviced?

3. From a group of 7 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if
    a. 2 particular men refuse to serve together?
    b. 2 particular women refuse to serve together?
    c. 1 particular man and 1 particular woman refuse to serve together?

4. Given all the start-up activity going on in high-tech, you realize that applying combinatorics to investment strategies might be an interesting idea to pursue. Say you have \$20 million that must be invested among 4 possible companies. Each investment must be in integral units of \$1 million, and there are minimal investments that need to be made if one is to invest in these companies. The minimal investments are \$1, \$2, \$3, and \$4 million dollars, respectively for company 1, 2, 3, and 4. How many different investment strategies are available if

a. an investment must be made in each company?
b. investments must be made in at least 3 of the 4 companies?

5. Determine the number of vectors ($x_1, x_2, \ldots, x_n$) such that each x_i is a non-negative integer and $\sum_{i=1}^{n} x_i \le k$, where k is some constant non-negative integer. Note that you can think of n (the size of the vector) as a constant that can be used in your answer.

6. In how many ways can n identical server requests ("identical balls") be distributed among r servers ("urns") so that the ith server receives at least m_i requests, for each $i = 1, 2, \ldots, r$? You can assume that $n \ge \sum_{i=1}^{r} m_i$.

7. If we assume that all possible poker hands (comprised of 5 cards from a standard 52 card deck) are equally likely, what is the probability of being dealt:
   a. a flush? (A hand is said to be a flush if all 5 cards are of the same suit. Note that this definition means that *straight flushes* (five cards of the same suit in numeric sequence) are also considered flushes.)
   b. one pair? (This occurs when the cards have numeric values a, a, b, c, d, where a, b, c, and d are all distinct.)
   c. two pairs? (This occurs when the cards have numeric values a, a, b, b, c, where a, b and c are all distinct.)
   d. three of a kind? (This occurs when the cards have numeric values a, a, a, b, c, where a, b and c are all distinct.)
   e. four of a kind? (This occurs when the cards have numeric values a, a, a, a, b.)

8. Say we roll a fair 6-sided die six times, what is the probability that:
   a. we will roll three different numbers, *twice* each?
   b. we will roll some number *exactly* 4 times?

9. Say we send out a total of 16 distinguishable emails to 10 distinct users, where each email we send is equally likely to go to any of the 10 users (note that it is possible that some users may not actually receive any email from us). What is the probability that the 16 emails are distributed such that there are 4 users who each receive exactly 2 emails each from us and 2 users who receive exactly 4 emails each from us?

10. To get good performance when working binary search trees (BST), we must consider the probability of producing completely degenerate BSTs (where each node in the BST has at most one child). See Handout #2, Example 7 for more details on binary search trees.
    a. If the integers 1 through n are inserted in arbitrary order into a BST (where each possible order is equally likely), what is the probability (as an expression in terms of n) that the resulting BST will have completely degenerate structure?
    b. Using your expression from part (a), determine the smallest value of n for which the probability of forming a completely degenerate BST is less than 0.01 (i.e., 1%).

11. Say a hacker has a list of n distinct password candidates, only one of which will successfully log her into a secure system.
    a. If she tries passwords from the list at random, deleting those passwords that do not work, what is the probability that her first successful login will be (exactly) on her k-th try?
    b. Now say the hacker tries passwords from the list at random, but does **not** delete previously tried passwords from the list. She stops after her first successful login attempt. What is the probability that her first successful login will be (exactly) on her k-th try?

12. Say a university is offering 3 programming classes: one in Java, one in C++, and one in Python. The classes are open to any of the 100 students at the university. There are:
    - a total of 27 students in the Java class
    - a total of 26 students in the C++ class
    - a total of 18 students in the Python class
    - 12 students in both the Java and C++ classes
    - 5 students in both the Java and Python classes
    - 7 students in both the C++ and Python classes
    - 3 students in all three classes (note: these students are also counted as being in each pair of classes in the numbers above).

    a. If a student is chosen randomly at the university, what is the probability that he or she is not in any of the 3 programming classes?
    b. If a student is chosen randomly at the university, what is the probability that he or she is taking *exactly one* of the three programming classes?
    c. If two students are chosen randomly at the university, what is the probability that at least one of the chosen students is taking at least one programming class?

13. A binary string containing M 0's and N 1's (in arbitrary order, where all orderings are equally likely) is sent over a network. What is the probability that the first r bits of the received message contain exactly k 1's?

14. A computer generates two random integers in the range 1 to 12, inclusive, where each value in the range 1 to 12 is equally likely to be generated. What is the probability that the second randomly generated integer has a value that is greater than the first?

15. Suppose that m strings are hashed (randomly) into N buckets, assuming that all N^m arrangements are equally likely. Find the probability that exactly k strings are hashed to the first bucket.